Roger Wolcott Richardson 1930–1993

G.I. Lehrer[1]

Roger Wolcott Richardson was born on 30 May 1930 in Baton Rouge, Louisiana. He was the eldest of the four sons of Roger Wolcott Richardson jr and Cora Johnson. His father was completing a Ph.D. in chemical engineering at the time of Richardson's birth and subsequently devoted virtually his whole working life to a career as an executive with Standard Oil (later the Exxon Corporation). His younger brother Hamilton was a well known U.S. Davis Cup tennis player. He had two other brothers, one of whom died of thyroid cancer at an early age. Richardson was educated at Baton Rouge High School where he was a champion athlete and tennis player and distinguished himself in mathematics and physics. He graduated with the degree of B.Sc., majoring in physics, from Louisiana State University in 1951 and was then conscripted into the U.S. Air Force. Richardson's work during this period apparently involved some mathematics and it later appeared to amuse him that it was secret, so that he could not tell his friends about it. He was posted to Cambridge Mass., and lived near the Harvard campus for two years. His desire to pursue an intellectual career was stimulated by the circle in which he moved during this period.

In September 1953 Richardson commenced graduate studies at Harvard University. There he met his contemporary Frank Raymond, who was to remain a lifelong friend. They were interested in geometry and topology, the latter of which was on the threshold of extending its influence far beyond the immediate circle of 'rubber sheet geometry' ideas which spawned it, to analysis, dynamical systems, algebra and even logic. However they found no permanent staff member at Harvard with those interests; moreover they found life at Harvard somewhat inimical. So, at the suggestion of Professor Gleason, they moved in May 1954 to the University of Michigan, Ann Arbor, which had a very high concentration of geometers and topologists at that time (see below). Their class was exceptional, containing several members who would soon rise to prominence, including the future Fields medallist Steve Smale. In 1958 Raymond and Richardson both completed Ph.D. degrees, Raymond under the supervision of the venerable topologist R.M. Wilder and Richardson under that of the well known Lie group theorist Hans Samelson. Raoul Bott, later to become a professor at Harvard, was at Ann Arbor as a junior staff member during this time.

The year 1958 was notable for Richardson in another respect. During his time at Ann Arbor he had met Margaret Jane Jewell Love ('Peggy' to her friends), who was doing postgraduate studies in English literature and in 1958 he married her. They remained inseparable.

In 1958 Richardson accepted an Instructorship at Princeton University. In Princeton there took place something of a reunion for his Ann Arbor class; James Munkres was also an instructor there, Raymond was a fellow at the Institute of Advanced Study, as was Smale. In Raymond's words, they were 'heady days for topologists' and the Institute and Princeton University were at the centre the topological world, with among others, Solomon Lefschetz at the University. During his period at Princeton, Richardson had a fruitful collaboration with E.E. Floyd (see below). In 1960 he accepted a tenure-track position at the University of Washington in Seattle, where he stayed until 1970. During this period he spent three separate years visiting centres for the study of group actions on topological or algebraic spaces: the Institute for Advanced Study at Princeton in 1963-64, Oxford in 1964-65, and Warwick (U.K.) in 1969–70.

In 1970, Richardson and his wife decided, for reasons related to the mood of the times in the U.S. (they were the 'Vietnam years'),

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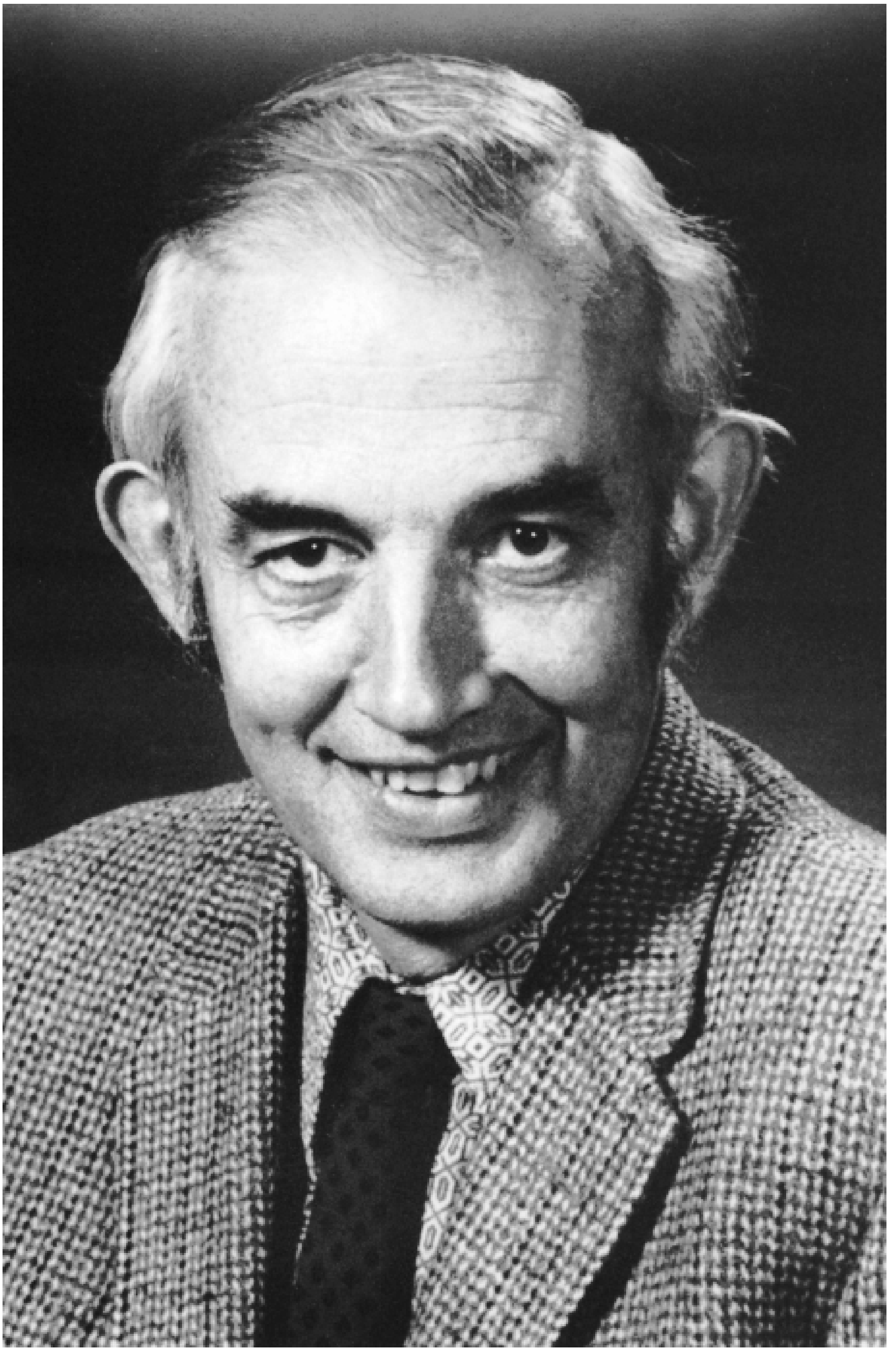

not to return to the U.S. He did not talk much about this decision, but he made it plain that he and Peggy felt that under Nixon, the country was adopting attitudes far to the right of what they could countenance. They had participated in the famous 'Washington moratorium' and march not long before their departure for the U.K., but clearly felt that the views promoted there would not prevail. Richardson accepted a chair at Durham University in the U.K. shortly afterwards. In 1978 he was appointed a Professorial Fellow in Mathematics at the Australian National University; in 1990 he was elected to the Australian Academy of Science and in 1992, his position at the A.N.U. was changed into a Chair in Mathematics, the post which he held until his death.

Although the Richardsons had no children themselves, they were very much involved with the families of their friends. Roger was particularly popular with children, since he was able to communicate with a direct simplicity which extended to his interactions with his adult peers. His opinions were clear and strong and although he generally was quite discreet about expressing them, when he did, there was no equivocation. He could therefore sometimes appear somewhat uncompromising, but this was generally no more than an instance of his shunning of the art of diplomacy. For these qualities he was widely respected in the world of his professional and academic peers, although they also meant that his life was not entirely free of conflict.

Richardson was involved in several research projects with collaborators around Australia and indeed the world. His death came at a time when he was actively using and pursuing reseach grants and when he was enjoying a particularly productive period. In February 1993, he had what was to have been a simple prostate operation, but immediately after the operation the news came that cancer had been discovered in his bladder. From that point, the news became progressively worse, until by the end of May it had transpired that his was a case of malignant lymphoma which had been particularly difficult to diagnose. He died on 15 June 1993.

Richardson's work had been concerned since the days of his thesis with different aspects of group actions on manifolds, especially the structure of the space of orbits. It might be considered as part of the mainstream of the 'Erlangen Program', established by Felix Klein, who defined geometry as the study of actions of groups of symmetries on spaces. Although the program had bifurcated into the continuous case, represented by Sophus Lie and his followers, and the discrete case pursued by Klein himself and his school, the two branches have now been to some extent unified. This unification is embodied in Roger Richardson's work. Some conception of the scope of this field may be conveyed by the observations that invariant theory may be regarded as one of its branches and that it may be approached from the point of view of topology, analysis or algebra. Two of this century's major trends in mathematics were the development of topology (or 'analysis situs') by means of attaching algebraic or numerical invariants to topological spaces and the algebraization of geometry far beyond anything which Descartes could have envisaged. In the latter, the properties of an algebraic variety (e.g. a curve) are studied by replacing it by a set of functions which characterize it completely. Richardson was involved in both movements. Early in his career at Ann Arbor he was surrounded by key figures such as Samelson, Raoul Bott and Steve Smale (who went on to win his Fields Medal for contributions to analytical aspects of 'orbit theory'). His work at this time revolved around symmetries and the structure of orbits of specific Lie groups acting on low dimensional topological spaces, such as spheres. He completely classified actions of the special orthogonal group $SO(3)$ and the symplectic group $Sp(1)$ on the four-dimensional sphere S^4 and related the corresponding actions on S^5 to the orbit structure.

In the early '60s, he developed an interest in algebraic geometry, and it was here that Richardson made his major impact. If X is an algebraic variety (e.g. a curve or surface) the symmetry properties of X may be approached by considering a group G of transformations of X; each element of G defines a bijective map from X to itself. Such a group G partitions X into orbits, which are themselves varieties of various dimensions. The closure of an orbit is the set of points which are infinitesimally near it; it is easy to see that such a closure must be a union of orbits. This concept gives rise to an intricate structure on the set of orbits. Richardson's best known result states that if P is a parabolic subgroup of a reductive group, then P has a dense orbit on its unipotent radical, i.e. one whose closure is the whole space. This orbit is now universally known as the 'Richardson orbit' and Richardson's theorem has been applied in many different ways. During his period in Australia Richardson had a fruitful collaboration with M.J. Field of Sydney on symmetry breaking; this work is described in some detail below. He also produced, with the distinguished algebraist T.A. Springer of Utrecht, a sequence of highly significant works on symmetric varieties associated to hermitian symmetric spaces.

Although Roger Richardson was not especially prolific, his papers are mature and polished and he took great care and time over them. They all contain interesting and sometimes striking results. Technicalities, with which he dealt with great competence, do not obscure the conceptual aspect of his work. His mathematical instinct or 'taste' was impeccable; his questions usually led to work of considerable depth.

From his position at the Australian National University, Richardson was able to identify and assist with the careers of many of the most promising young mathematicians in the country. Many of them testify to the open-minded frankness with which he approached any issue. He was one of the founders of the annual Australian Lie Group Conference, the first meeting of which took place in 1989.

The works of Richardson are listed in the Bibliography at the end of this Memoir and are referred to by number. All other references are by code to the 'References' which may also be found below.

Virtually all of Richardson's work involves the concept of symmetries of some geometric space, which he interpreted in the sense of the 'Erlangen Program', *viz.* in terms of group actions on various spaces. For most of his career, this involved studying the subtle interplay between group actions, geometric properties of the space concerned and the structure and properties of various spaces of functions on the underlying geometric space. However during the last five or six years of his life, he collaborated with M.J. Field in a series of important papers on equivariant bifurcation theory, in which ideas from the mathematical world in which he normally worked were applied to dynamical systems. This work is described separately below.

A. Geometry and Group Actions

Symmetries of spheres

The problem given to Richardson by H. Samelson for his thesis involved group actions on spheres. An n-dimensional sphere S^n may be thought of as the set of unit vectors in an $(n+1)$–dimensional Euclidean space. Hence the orthogonal group $O\ (n+1)$ acts (linearly) on S^n. This action describes the 'obvious' symmetries of S^n. The question addressed by Richardson in his thesis is whether there are other symmetries-i.e. group actions on S^n not 'equivalent' to the linear action of (a subgroup of) $O(n + 1)$. He discovered many and these form the subject of his publications [3] and [4], with [2] addressing the same subject. This early work was topological in nature, as was his first published paper [1] with E.E. Floyd, which deals with the subtle question as to whether a finite group acting on a cell always has a fixed point. The paper gives a counter-example of the alternating group A_5 acting on a high dimensional cell. This work was the result of a collaboration with Floyd during his Princeton days.

Deformation theory of Lie algebras

After this early topological work, Richardson had moved to the University of Washington in Seattle and there had a long collaboration with A. Nijenhuis on the deformation theory of algebraic structures. In this theory, one studies algebraic structures which come in parametrized families and whose structure usually varies 'continuously' with the parameter. For example if an associative algebra A is generated over the real numbers $\mathbb{R}$ by a single generator x, such that $x^2 = (t^2–1)x$ (t a parameter) then A is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ unless $t = \pm1$, in which case A is isomorphic to an algebra of triangular matrices. More generally, one might consider any associative or Lie algebra whose structure constants (multiplication table) depend on parameters. If these parameters are thought of as varying over a space P (above it is $\mathbb{R}$) one might study this situation geometrically by considering the map $P \longrightarrow X$, where X is a 'space of algebras'. This may be done in an algebraic or analytic context, according to whether the spaces are thought of as algebraic or analytic (C^∞, complex analytic, etc.) varieties. A basic question is whether the structure of algebras with similar parameters is similar; e.g. are they isomorphic? In the case of Lie algebras, Nijenhuis and Richardson showed in [6] and [7] that there are cohomological conditions for this to be true.

The above problems are similar to questions in the theory of deformations of complex structures which had been developed by Kodaira, Nirenberg and Spencer [KNS] and by Kuranishi [Ki] some years earlier; this possibly influenced Nijenhuis and Richardson. Their approach involved the detailed study of tangent spaces. Such methods became a hallmark of much of Richardson's subsequent work on a variety of questions. He himself referred to them as his 'specialty'.

The first case in point is his celebrated work [8], which addresses the question: are there finitely many unipotent conjugacy classes in a semi-simple algebraic group? In the case of the linear groups, one has the standard 'Jordan form' for matrices, which parameterizes unipotent classes by partitions, valid over any field. Kostant and Dynkin (see [K]) had answered the question (concerning the finiteness of the number of unipotent classes) affirmatively for complex groups. When the underlying field is arbitrary, Richardson also gave, in [8], an affirmative answer with some restriction on the characteristic which was necessitated by the proof, which uses a tangent space computation to reduce the question to the theory of Jordan forms. The general case, including the cases not covered by Richardson's theory, was later settled by Lusztig [L] using highly indirect methods, including his classification with Deligne [DL] of the irreducible characters of the corresponding finite groups of rational points. Richardson's remains the most conceptual approach, although it does not yield the most general result. To illustrate the geometric flavour of deformation theoretic questions, we describe the main result of [10]. Let $\mathfrak{L}$ be a Lie algebra over the complex numbers $\mathbb{C}$. The subalgebras $\mathfrak{M}$ of $\mathfrak{L}$ which have a given dimension n form a projective algebraic variety X, which is a closed subvariety of the classical Grassmanian variety of all n-dimensional linear subspaces of $\mathfrak{L}$. The group G of all automorphisms of $\mathfrak{L}$ acts on X and a subalgebra $\mathfrak{M} \in X$ is called rigid if its orbit $G.\mathfrak{M}$ under the action of G on X is open in X (in the Zariski topology). Intuitively, this means that with 'probability one' any n-dimensional subalgebra is conjugate to $\mathfrak{M}$. Moreover every n-dimensional subalgebra is 'a deformation of' $\mathfrak{M}$. Richardson showed in [10] that $\mathfrak{M}$ is rigid if the Lie algebra cohomology group $H^1(\mathfrak{M}, \mathfrak{L}/\mathfrak{M})$ vanishes, an 'internal' algebraic condition.

Generic Isotropy Groups

Following his work on deformations of Lie algebras, Richardson turned to the corresponding questions for Lie or algebraic groups, which is somewhat more delicate. In the work [9], Richardson took up the subject of deformations of Lie subgroups of a Lie group and proved the following result, which is in the spirit of that described in the previous paragraph. Let G be a real or complex Lie group and let $(H_x)_{x\in X}$ be a collection of subgroups of G which are parameterized by a manifold X. A typical situation where this might arise is where G acts on a manifold (or variety) X and one takes H_x to be the isotropy group of $x \in X$ (i.e. $H_x = G_x = \{g \in G \mid g \cdot x = x\}$). The key question addressed in [19] is: how does (the structure etc. of) Hx vary with x? In the real analytic context, the following is proved in [19]. Let $K = H_{x0}$ be one of the groups in the collection and suppose that the component group K/K^0 is finitely generated. If H^1 (K, Lie (G)/Lie (K)) = 0, then there is an open neighbourhood U of x_0 in X and an analytic map $h : U \longrightarrow G$ such that $\eta(x_0) = e$ (the identity element) and such that for all $x \in U$, $\eta(x)\, H_{x_0}\, \eta(x)^{-1}$ is a subgroup of K. In the prototypical case where G acts on X and the H_x are the isotropy groups G_x, G acts on the collection $\{H_x\}_{x\in X}$ by conjugation and one has the concept of rigidity as above; the stated result then becomes an assertion of rigidity. These questions and results had led to the notion of 'generic isotropy group' which had arisen earlier in the work of Montgomery and Zippin.

They had shown that if a compact Lie group G acts on a manifold X, then there is an open and dense submanifold U of X such that for $x \in U$, the isotropy groups G_x all lie in one conjugacy class (of subgroups of G). Thus one speaks of the 'generic' isotropy group. Richardson addressed the question of generic isotropy groups (or 'principal orbit types', following an older terminology) in the context of more general group actions in [20]. If G is not compact, generic isotropy groups need not exist; examples of this, which involve non-reductive groups G, had been known earlier. In his paper [20], the significance of which will be explained below, Richardson proved the following result concerning generic isotropy groups.

Let G be a linear algebraic group over $\mathbb{C}$ which acts on a smooth irreducible affine variety X. Suppose $x \in X$ is such that G_x is reductive. Then there is an open (Zariski) neighbourhood U of x such that for $y \in U$, the identity component G_y^0 is conjugate to a subgroup of G_x. In [21], [22] and [24] he discusses the various subtleties which arise in considering the existence of generic isotropy groups in various contexts, including real analytic actions of real reductive Lie groups and complex analytic actions of complex reductive Lie groups on complex or Stein manifolds.

The Slice Theorem and Applications to Orbit Structure

The significance of [20] extends further than the result itself. Shortly after its appearance in 1972, D. Luna published his 'slice theorem' in [Lu1]. Suppose G is an affine algebraic group acting on a variety X. Let $x \in X$ be a point such that the orbit $G \cdot x$ is closed (for example any orbit of minimal dimension will do). Then Luna's slice theorem asserts roughly that the orbit $G \cdot x$ has an *étale* neighbourhood N which is a fibre product over G_x of G, with a subvariety S of X. Here 'étale neighbourhood' means neighbourhood in the étale topology, where open sets are replaced by morphisms $\phi : U \longrightarrow X$ which induce isomorphisms of the tangent spaces. This means that the neighbourhood N is a union of G-orbits and N is isomorphic to $G \times_{G_x} S$ (fibre product), which is a fibre space over G/G_x ($\cong G \cdot x$) with fibre S. Intuitively, N is rather like a 'product' of the orbit $G \cdot x$ and the 'slice' S.

The slice theorem has acquired fundamental importance in the theory of algebraic group actions (see [S]). It implies the result of [20]. In [29], Richardson and Luna gave another application which puts the Chevalley restriction theorem (see [B]) into a general setting. Their result asserts that if X (above) is normal ('almost' smooth) and $Y \subseteq X$ is the set of fixed points of G_x, then we have an isomorphism of orbit spaces : $Y/\!/W \xrightarrow{\sim} X/\!/G$, where $W = N_G(G_x)/G_x$ is the normaliser of G_x modulo G_x. Chevalley's theorem deals with the conjugation action of a reductive group on itself. In that case G_x becomes a maximal torus and W is the Weyl group. In [35], Richardson and Bardsley established a 'positive characteristic version' of the slice theorem, and gave various applications.

It is in the paper [23] that Richardson discovered the famous 'Richardson orbit'. Let P be a proper parabolic subgroup of a semi-simple algebraic group G and write U for its unipotent radical. Richardson's result is that P, which acts on U since U is normal in P, has a dense open orbit in U. The G-conjugacy class of the elements of this orbit is called the Richardson class of G which corresponds to P. These classes are the key concept used by Bala and Carter [BC1, BC2] in giving a complete (generic) classification of the unipotent conjugacy classes of reductive groups. This is the classification now used in most applications. Richardson orbits have been used by Lusztig and Spaltenstein [LS] to define an 'induction' process for unipotent classes and they also appear in the theory of primitive ideals in enveloping algebras; they are important as well in the character theory of reductive groups over finite fields. In the work [DLM], Richardson made an important contribution in this context.

Unipotent orbits also appear in the paper [28], in which Richardson showed that the variety of all pairs of commuting elements in a semi-simple, simply connected algebraic group G over $\mathbb{C}$ is irreducible and similarly for its Lie algebra. His proof shows that by induction, it suffices to consider the case where one of the elements is unipotent. The delicate question as to whether this variety is reduced was known to Richardson, but remains unresolved.

Groups with Involutions, Orbit Theory

Richardson next turned to the study of algebraic groups with involutions, the prototype of the situation being the case of a Cartan involution of a complex Lie group, whose group of fixed points is a maximal compact subgroup, an example being the group of unitary matrices as a subgroup of $GL_n(\mathbb{C})$. In general, let G be a connected reductive algebraic group over an algebraically closed field of characteristic not equal to 2. Let θ be an (algebraic) automorphism of G of order 2. An example other than the prototype is the 'trivial' one, where G is a product $H \times H$ (H reductive) and θ permutes the factors. Let $K = G^\theta$ be the group of fixed points of θ; this is a closed reductive subgroup of G and the quotient $S = G/K$ is an affine variety on which G

(and therefore a *fortiori K*) acts by left translations. The K-action on S is the subject of the paper [34]. In the 'trivial' case (see above) this amounts to the study of the conjugation action of H on itself, since in that case, K is just H, realised as the diagonal subgroup of $H \times H$. Thus the results of [34] generalise those on conjugacy classes of reductive groups, e.g. in [SpSt]. These results also extend those of Kostant and Rallis [KR], who studied the linear action of K on the (–1)-eigenspace of θ on the Lie algebra of G.

Still following the above theme, Richardson's paper [37] on simultaneous conjugacy, draws on results from his previous papers [29], [32] and [8] and addresses problems of some delicacy. Among his results is the establishment of a type of 'Jordan decomposition' for elements of S = G/K into semisimple and unipotent parts; in analogy with the group case, he shows that the closed K-orbits in S are the semi-simple ones and that the variety of closed orbits is affine. This paper ([37]) is also of interest for the analysis of real Lie group actions on symmetric spaces; the paper provides insights into the geometry of the complexifications of these spaces. Richardson mentioned to T.A. Springer some time before his death that he had given some thought to a generalization of the above situation where one has two commuting involutions θ_1 and θ_2 on G, with corresponding fixed point subgroups K_1 and K_2; in particular he asserted that results like those of [37] were true in this more general situation, which would be of some significance because of its relevance to the algebraic geometry of 'affine symmetric spaces' of real Lie groups.

The paper [45] with Springer deals with the action of a Borel subgroup B of G on the symmetric variety $S = G/K$ (as above). It was known that B generally has finitely many orbits on S; the (finite) set V of orbits is partially ordered under the usual closure relation on orbits: the closure of a given orbit is a union of orbits, which are decreed to be smaller. In the 'trivial' case, V may be identified with the Weyl group W of H and the order relation is the Bruhat (or Chevalley-Bruhat) order on W. In [45] corresponding results are proved in general for V. In his posthumous paper [53], Richardson elaborates on particular cases of this work for various of the classical groups.

In [50], G. Röhrle and R. Steinberg combined forces with Richardson to address the question, raised by G. Seitz, of classifying the (finite set of) orbits of a parabolic subgroup P of G on its unipotent radical in case that radical is abelian. Let K be a Levi subgroup of P. Then the classification of the P-orbits on its unipotent radical is equivalent to the classification of the P-orbits on $S = G/K$, which fits into the context above. The result is an elegant solution which involves just root data; involved in this work one again finds Richardson's favourite tangent space computations.

Geometric Invariant Theory

Geometric invariant theory is the theme of Richardson's papers [38], [39], [40], [42] and [46]. One might describe the general area as follows: if X is a complex algebraic variety and $\mathbb{C}[X]$ is the algebra of regular functions on X, let H be a group acting on X. One obtains functions on the space X/H of orbits of H on X from those functions in $\mathbb{C}[X]$ which are invariant under H. Thus the study of orbit spaces may be interpreted as the study of the algebra $\mathbb{C}[X]^H$ of invariants of a group acting on an algebra. His important paper [39] deals with the following question. Let G be a connected semi-simple complex Lie group, with Lie algebra $\mathfrak{G}$. If X is a G-stable subvariety of $\mathfrak{G}$, regarded as an affine G-space, when is X a normal (i.e. almost smooth) variety? Richardson's criterion is couched in terms of invariants of a certain Weyl group action. Let Y be a Cartan subalgebra of $\mathfrak{G}$ and write W for the Weyl group of $\mathfrak{G}$ with respect to Y. Let D be an irreducible component of the intersection $X \cap Y$. Then W acts on $X \cap Y$, permuting its irreducible components. If W_0 is the stabilizer in W of D then we may speak of the algebras $\mathbb{C}[Y]^W$ and $\mathbb{C}[D]^{W_0}$ of W-(respectively W_0)-invariant regular functions on Y (respectively D). One then has the restriction map : $\mathbb{C}[Y]^W \to \mathbb{C}[D]^{W_0}$ and the necessary condition given by Richardson in [39] for the normality of X is that this restriction map should be surjective. This criterion is applied to several examples. One of these occurs in the context of Kostant and Rallis: one takes X to be the closure of $Ad(G).E$, where E is the (-1)-eigenspace of an involutory automorphism of $\mathfrak{G}$ (see above). The case which has possibly led to the most significant examples is where X is the closure of a 'decomposition class' in $\mathfrak{G}$, a decomposition class being a subvariety of elements with similar Jordan decomposition. In both these cases counterexamples to the assertion of normality are found. A particular example of a decomposition class is a unipotent orbit in $\mathfrak{G}$.

The question of the normality of unipotent orbit closures is a classical one and is addressed in [38]. Take $\mathfrak{G}$ to be the algebra above; let $P_1, \cdots, P_\ell$ be algebraically independent homogeneous generators of $\mathbb{C}[\mathfrak{G}]^G$ and let $\pi : \mathfrak{G} \longrightarrow \mathbb{C}^\ell$ be defined by $\pi(x) = (P_1(x), \cdots, P_\ell(x))$. In the case where $\mathfrak{G}$ is of 'type A', π is just the map which takes a matrix to its characteristic polynomial ($\pm P_i(x)$ being the coefficients). The chief result of [38] is the determination of the rank of the derivative $d\pi_x$ at any $x \in \mathfrak{G}$. This result is used to study the normality of orbit closures as follows. Define a sheet (see [BK]) in $\mathfrak{G}$ to be an irreducible component of the (locally closed) subvariety of G consisting of elements whose centraliser in G has a given fixed dimension. Let $\boldsymbol{S}$ be a sheet; then $\boldsymbol{S}$ contains a unique unipotent orbit, say $\boldsymbol{O} = Ad(G) \cdot x$. If x has connected centraliser and if the closure of $\boldsymbol{O}$ is normal, Richardson shows that the function $y \mapsto \text{rank } d\pi_y$ is constant on $\boldsymbol{S}$. This result may be used to find several examples non-normal closures of unipotent orbits, solving problems of some subtlety.

Equivariant Bifurcation Theory

Contributed by Professor Ian Stewart

Beginning in the late 1980s, working jointly with Mike Field, Roger Richardson made a remarkable series of deep discoveries about bifurcation in systems of differential equations with symmetry. Their work shed a great deal of light on puzzling phenomena involving symmetry-breaking, and has stimulated many further developments. The main papers are Field and Richardson [41, 47, 49], together with an announcement in [44].

Suppose that $X = \mathbb{R}^n$ and let G be a compact Lie group acting linearly (and without loss of generality orthogonally) on X. A *bifurcation problem* is a mapping

$$g : X \times \mathbb{R} \to X$$

$$(x, \lambda) \mapsto g(x, \lambda)$$

and we say that g is *G-equivariant* if

$$g(\gamma.x,\lambda) = \gamma.g(x, \lambda)$$

for all $\gamma \varepsilon G$. Usually the mapping g is assumed to be smooth (that is, of class C^∞).

The phrase 'bifurcation problem' arises in the context of a differential equation of the form

$$\frac{dx}{dt} = g(x, \lambda)$$

in which λ is a parameter, known as the *bifurcation parameter*. The main question about such systems of differential equations is: how do solutions $x = x(t)$ change as λ varies? Any value of λ that corresponds to a local change in the topological type of the phase portrait of the solutions is said to be a *bifurcation point*, and the aim of the theory is to understand how the solutions x change near specific types of bifurcation point. The simplest case, and one that is central to the theory, is when we seek steady states, meaning that x remains constant. Then $\frac{dx}{dt} = 0$, so we must find the zeros of g, given by $g(x,\lambda) = 0$.

Maximal Isotropy Subgroup Conjecture

An important phenomenon is *spontaneous symmetry-breaking*, in which solutions possess less symmetry than the entire group G. Specifically, if we define the *isotropy subgroup* of a point $x \in X$ to be $G_x = \{\gamma \in G \mid \gamma.x = x\}$ as above, then it may happen that the equation $g(x,\lambda) = 0$ has a solution x for which $G_x \neq G$. In such a case we say that x breaks the symmetry G. For example, suppose that $n = 1$ and that $g(x,\lambda) = x^3 - \lambda x$. This is equivariant for the action of $G = \mathbb{Z}_2 = \{id, \sigma\}$ in which $\sigma.x = -x$. Solutions of $g = 0$ are $x = 0$ for all λ, together with $x = \pm\sqrt{\lambda}$ when $\lambda > 0$. The latter solutions break symmetry since they have trivial isotropy. They are said to form 'branches' because they are curves parametrized by λ.

Until recently a key conjecture in the area was the Maximal Isotropy Subgroup Conjecture (MISC), which states that if the group G acts absolutely irreducibly on X, then generically each nontrivial branch of zeros of g corresponds to a maximal isotropy subgroup. Moreover, all maximal isotropy subgroups can be realised by selecting a suitable g. This conjecture has proved influential and useful–even though it is false. Counterexamples were found by Chossat [Ch] and Lauterbach [La], and in the context of the Landau theory of phase transitions other counterexamples were found by Jaric [Ja] and by Mukamel and Jaric [MJa].

In [41], Field and Richardson describe a whole class of counterexamples, in a context that makes it clear why they are counterexamples. They achieve this by embedding the problem in a rich area of mathematics, the theory of finite real reflection groups (otherwise known as finite Coxeter groups). They first characterize symmetry-breaking and the MISC when the group G is a Weyl group $W(\mathfrak{L})$ of a simple complex Lie algebra $\mathfrak{L}$. Specifically, they show that the MISC is valid for the groups $W(A_\kappa)$, $W(B_\kappa)$, $W(F_4)$ as well as for the dihedral groups $I_2(p)$ and the icosahedral group H_3. In contrast, they show that the MISC *fails* for $W(D_\kappa), \kappa \geq 4$. The results also carry over to the adjoint representation of the corresponding semisimple Lie group. In particular the MISC holds for the adjoint representations of the special unitary groups $SU(n)$ and the odd-dimensional special orthogonal groups $SO(2n+1)$, but fails for the even-dimensional special orthogonal groups $SO(2n)$.

Branching Theory

This work is taken further in [47] and [49]. The first paper develops a number of techniques for analysing branching behaviour in bifurcation problems, and the second applies them to produce a number of examples where bifurcation to branches with submaximal isotropy occurs. Indeed the authors provide evidence for the view that this should be anticipated as a relatively common phenomenon. In short, in general the MISC does not provide even a rough 'rule of thumb' guide to what kind of symmetry-breaking is likely to occur. Not only does it fail: it fails spectacularly.

The paper [47] is largely concerned with obtaining good determinacy criteria for bifurcation problems. A bifurcation problem g is said to be *κ-determined* if it is equivalent (in a sense that preserves the bifurcation structure and any relevant symmetries) to its Taylor series truncated at degree κ. The main technical theorem is a result on the local stability of branching, which permits a key reduction in the study of many bifurcation problems–namely the elimination of the bifurcation parameter, so that the problem can be studied in terms of parameter-free equivariant vector fields on the unit sphere.

The second paper [49] applies this machinery to many specific examples. The determinacy criteria are used to show that $W = W(B_\kappa)$ is 3-determined in its natural action on $\mathbb{R}^\kappa$, that is, that generically the topology and isotropy of its branches may be obtained by truncating the map q at cubic order. This immediately reduces the whole question to a specific, concrete system of equations. Moreover, the maximal isotropy subgroups of W are determined by the symmetry axes of the κ-dimensional cube.

Now suppose that G is a subgroup of W that acts absolutely irreducibly on $\mathbb{R}^\kappa$ and that there are no nontrivial quadratic G-equivariants. Then Field and Richardson prove the remarkable result that G is also 3-determined. This has immediate implications, because some solution branches for G-equivariant problems can now be read off by determining the isotropy subgroups in G of symmetry axes of the κ-cube. Often–indeed usually–these are no longer maximal isotropy subgroups in G. One important application of this work is to the bifurcation of stable heteroclinic cycles, which are formed from a system of saddlepoint equilibria when the unstable manifold of one saddle connects to the stable manifold of the next saddle in the cycle. For symmetric systems, such cycles can be predicted on the basis of suitable group-theoretic criteria. Paper [49] provides a number of examples where generic bifurcation to heteroclinic cycles, or to periodic orbits, occurs.

From Richardson's point of view, equivariant bifurcation theory was a particular case of the general questions about group actions and their orbits which were the underlying theme of all his work; his papers with Field are regarded as among the deepest in the whole field of equivariant bifurcation theory.

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